Comment II on "Instability threshold in the Bénard-Marangoni problem"

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The classical theory of surface-tension-driven convection by Pearson [J. Fluid Mech. **4**, 489 (1958)] has been challenged by Rabin [Phys. Rev. E **53**, R2057 (1996)] on the grounds that Pearson used an improper thermal boundary condition at the upper surface of the liquid. We show that Pearson's theory is correct. [S1063-651X(97)11910-9]

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In 1958, Pearson [1] published a theory of thermocapillary convection that explained how surface-tension gradients due to thermal gradients along a liquid-gas interface lead to the onset of Bénard convection. Pearson's work has been repeated and expanded upon by numerous authors [2–6] and confirmed in laboratory experiments [7]. Recently, however, Rabin [8] argued that Pearson used an improper thermal boundary condition at the liquid-gas interface and thus Pearson's condition for instability is incorrect. Pearson had calculated the minimum critical Marangoni number M_c to be 80. Rabin calculated (a differently defined) M_c to be 222.

In this Comment, we point out three basic problems in Rabin's analysis. First, Rabin employed a different definition of M than did Pearson. Much of the difference in onset predictions between Pearson and Rabin can be attributed to this difference in definition, which is not explicitly stated in [8]. Second, Rabin ignored the wave number dependence of the Biot number by using the total temperature, not the perturbation temperature, in his upper thermal boundary condition. Finally, Rabin treated the the Biot number as a free parameter. This cannot be done. These last two assumptions in combination have nonphysical implications. We will discuss the problem in terms of a liquid layer with a gas layer above it; the liquid-gas system is contained between two horizontal, thermally conducting plates. This geometry is used in most Bénard convection experiments [7,9,10].

Rabin employed a definition of the Marangoni number M that was based on the total temperature drop ΔT_{total} across the system (i.e., the temperature drop across both liquid and gas layers). He then calculated the critical M_c at which the system becomes unstable to infinitesimal perturbations of wave number k [Eq. (10) in [8]]:

$$M_c(k)=\frac{(\text{Bi}+1)(\text{Bi}\ \ \sinh k+k\cosh k)}{\text{Bi}}f(k), \tag{1}$$

$$f(k)=\frac{4k(\sinh 2k-2k)}{(\sinh k^3-k^3\cosh k)}, \tag{2}$$

where Bi is a Biot number that describes heat transfer at the liquid-gas interface. Pearson and most subsequent authors employed a Marangoni number based on the temperature drop ΔT across the liquid layer only. From Fourier's law of heat conduction [9],

$$\Delta T_{\text{total}}=\Delta T\left(1+\frac{k_l d_g}{k_g d_l}\right)=\Delta T\left(\frac{1+\text{Bi}_{k=0}}{\text{Bi}_{k=0}}\right), \tag{3}$$

where d_l, d_g, k_l, k_g are the depths and thermal conductivities of the liquid and gas layers and $\text{Bi}_{k=0}\equiv k_g d_l/k_l d_g$ is the Biot number for the conduction state. The prediction of Pearson for the onset of instability (using Rabin's definition of M with ΔT_{total}) is then

$$M_c(k)=\frac{(\text{Bi}_{k=0}+1)[\text{Bi}(k)\sinh k+k\cosh k]}{\text{Bi}_{k=0}}f(k), \tag{4}$$

where $\text{Bi}(k)$ is the Biot number for the perturbation (Pearson's L) and the typographical errors in Pearson's Eq. (27) have been corrected (see [2]).

Equations (1) and (4) are identical except that the Pearson result (4) has two Biot numbers (one for the conduction state and one for the perturbation) while the Rabin result (1) has only one Biot number (for both the conduction state and the perturbation). Thus, if the Biot numbers for the conduction state and for the perturbation are identical, then Rabin agrees exactly with Pearson; if the two Biot numbers differ, then Rabin's equation is in error. In general, $\text{Bi}(k)\neq\text{Bi}_{k=0}$ since for an undeformed interface (the situation considered by both Pearson and Rabin) the Biot number is [11,12]

$$\text{Bi}(k)=kd_l\frac{k_g}{k_l}\coth kd_g\,. \tag{5}$$

Rabin's single Biot number results from using a single thermal boundary condition at the liquid-gas interface, rather than a separate condition for the conduction and perturbation temperatures. The single Biot number (or thermal boundary condition) ignores the wave number dependence of heat transfer. In many cases, the Biot number correction is small and the wave number dependence is even smaller [e.g., for equally thick layers of silicone oil and air, the wave number dependence of $\text{Bi}(k)$ shifts the minimum from $k=1.99$ to $k=1.98$]. Because the difference is small, the distinction between the two Biot numbers has not been made in some experimental papers [7]; a theoretical analysis, however, should preserve the distinction.

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A more fundamental problem with Rabin's calculation arises from his assumption that the Biot number is a free parameter that the system selects. To obtain a minimum of $M_c = 222$ (for $k = 2.33$ and $\mathrm{Bi} = 1.54$), Rabin searched for the global minimum of M_c in (k,Bi) space. However, since $\mathrm{Bi} = \mathrm{Bi}(k;d_l,d_g,k_l,k_g)$, Bi cannot be considered a free parameter. Since Rabin used the same Biot number for the conduction state and for the perturbation, his treatment of Bi as a free parameter implies that the conduction (unperturbed) state is undetermined until the system is perturbed—i.e., the perturbation selects the (prior) unperturbed state.

Finally, Rabin does not cite any of the sizable volume of theoretical work that has been done on the Marangoni problem since 1964 (e.g., [3–6,13,14]). For the case of a one-layer model with zero Rayleigh number (no buoyancy effect) and an undeformed interface, Pearson's result still stands.

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[1] J. R. A. Pearson, J. Fluid Mech. **4**, 489 (1958).
[2] D. A. Nield, J. Fluid Mech. **19**, 341 (1964).
[3] K. A. Smith, J. Fluid Mech. **24**, 401 (1966).
[4] M. Takashima, J. Phys. Soc. Jpn. **50**, 2745 (1981).
[5] D. A. Goussis and R. E. Kelly, Int. J. Heat Mass Transf. **33**, 2237 (1990).
[6] C. Pérez-García and G. Carneiro, Phys. Fluids A **3**, 292 (1991).
[7] M. F. Schatz, S. J. VanHook, W. D. McCormick, J. B. Swift, and H. L. Swinney, Phys. Rev. Lett. **75**, 1938 (1995). The authors used the conduction Biot number for the perturbation and stated a theoretical prediction of $M_c = 87$ (Pearson's definition of M). Using the appropriate Biot number for a $k = 2$ perturbation, the theoretical prediction for onset would be $M_c = 95$. The authors measured $M_c = 84 \pm 11$, where the uncertainty is due to uncertainty in fluid parameters.
[8] L. M. Rabin, Phys. Rev. E **53**, R2057 (1996).
[9] E. L. Koschmieder and M. I. Biggerstaff, J. Fluid Mech. **167**, 49 (1986).
[10] S. J. VanHook, M. F. Schatz, W. D. McCormick, J. B. Swift, and H. L. Swinney, Phys. Rev. Lett. **75**, 4397 (1995).
[11] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[12] L. Hadji, J. Safar, and M. Schell, J. Non-Equilib. Thermodyn. **16**, 343 (1991). In the Appendix, the authors calculate the Biot number for the more general case of top and bottom plates with finite thermal conductivity.
[13] S. H. Davis, Annu. Rev. Fluid Mech. **19**, 403 (1987).
[14] A. A. Golovin, A. A. Nepomnyashchy, and L. M. Pismen, Phys. Fluids **6**, 34 (1994).